

# ON SOME STRONG CONVERGENCE RESULTS OF A NEW HALPERN-TYPE ITERATIVE PROCESS FOR QUASI-NONEXPANSIVE MAPPINGS AND ACCRETIVE OPERATORS IN BANACH SPACES

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**ABSTRACT.** In this study, we introduce a new iterative processes to approximate common fixed points of an infinite family of quasi-nonexpansive mappings and obtain a strongly convergent iterative sequence to the common fixed points of these mappings in a uniformly convex Banach space. Also we prove that this process approximates to zeros of an infinite family of accretive operators and we obtain a strong convergence result for these operators.

## 1. INTRODUCTION AND PRELIMINARIES

Throughout this study, the set of all non-negative integers and the set of real numbers, which we denote by  $\mathbb{N}$  and  $\mathbb{R}$ , respectively.

Geometric properties of Banach spaces and nonlinear algorithms, a topic of intensive research efforts, in particular within the past 30 years, or so. Some geometric properties of Banach spaces play a crucial role in fixed point theory. In the first part of the study, we investigate these geometric concepts most of which are well known. We begin with some basic notations.

In 1936, Clarkson [1] achieved a remarkable study on uniform convexity. It signalled the beginning of extensive research efforts on the geometry of Banach spaces and its applications. Most of the results indicated in this work were developed in 1991 or later.

Let  $C$  be a nonempty, closed and convex set, which is subset of  $B$  Banach space, and let  $B^*$  be the dual space of  $B$ . We define the modulus of convexity of  $B$ ,  $\delta_B(\epsilon)$ , as follows:

$$\delta_B(\epsilon) = \inf \left\{ 1 - \frac{\|a+b\|}{2} : a, b \in \overline{B(0,1)}, \|a-b\| \geq \epsilon \right\}.$$

The modulus of convexity is a real valued function defined from  $[0, 2]$  to  $[0, 1]$  which is continuous on  $[0, 2)$ . A Banach space is uniformly convex if and only if  $\delta_B(\epsilon) > 0$  for all  $\epsilon > 0$ . Let  $B$  be a normed space and  $S_B = \{a \in B : \|a\| = 1\}$  the unit sphere of  $B$ . Then norm of  $B$  is Gâteaux differentiable at point  $a \in S_B$  if

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for  $a \in S_B$

$$\frac{d}{dt} (\|a + tb\|) |_{t=0} = \lim_{t \rightarrow 0} \frac{\|a + tb\| - \|a\|}{t}$$

exists. The norm of  $B$  is said to Gâteaux differentiable if it is Gâteaux differentiable at each point of  $S_B$ . In the case,  $B$  is called smooth. The norm of  $B$  is said to uniformly Gâteaux differentiable if for each  $b \in S_B$ , the limit is approached uniformly for  $a \in S_B$ . Similarly, if the norm of  $B$  is uniformly Gâteaux differentiable, then  $B$  is called uniformly smooth. A normed space  $B$  is called strictly convex if for all  $a, b \in B$ ,  $a \neq b$ ,  $\|a\| = \|b\| = 1$ , we have

$$\|\lambda a + (1 - \lambda) b\| < 1, \quad \text{for all } \lambda \in (0, 1).$$

Now, the result of the above definitions we give the following theorem and corollary without proofs.

**Theorem 1.** [2] *Let  $B$  be a Banach space.*

- 1)  *$B$  is uniformly convex if and only if  $B^*$  is uniformly smooth.*
- 2)  *$B$  is uniformly smooth if and only if  $B^*$  is uniformly smooth.*

**Theorem 2.** [2] *Every uniformly smooth space is reflexive.*

A self mapping  $\phi$  on  $[0, \infty)$  is said to be a gauge map if it is continuous and strictly increasing such that  $\phi(0) = 0$ . Let  $\phi$  be a gauge function, and let  $B$  be any normed space. If the mapping  $J_\phi : B \rightarrow 2^{B^*}$  defined by

$$J_\phi a = \{f \in B^* : \langle a, f \rangle = \|a\| \|f\|; \|f\| = \phi(\|a\|)\}$$

for all  $a \in B$ , then  $J_\phi$  is said to be the duality map with gauge function  $\phi$ . If  $\phi(t) = t$  is selected, then  $J_\phi = J$  duality mapping is called the normalized duality map.

Let

$$\psi(t) = \int_0^t \phi(s) ds, \quad t \geq 0,$$

then  $\psi(\delta t) \leq \delta \phi(t)$  for each  $\delta \in (0, 1)$ .

$$\rho(t) = \sup \left\{ \frac{\|a + b\| + \|a - b\|}{2} - 1 : a, b \in B, \|a\| = 1 \text{ and } \|b\| = t \right\}$$

is called the modulus of smoothness of  $B$ , where  $\rho : [0, \infty) \rightarrow [0, \infty)$  is a mapping. Also,  $\lim_{t \rightarrow 0} \frac{\rho(t)}{t} = 0$  if and only if  $B$  is uniformly smoothness.

Assume that  $q \in \mathbb{R}$  is chosen in the interval  $(1, 2]$ . If a Banach space  $B$  is  $q$ -uniformly smoothness, then it provides the following conditions. (i) there exists a fix  $c > 0$ , (ii)  $\rho(t) \leq ct^q$ . For  $q > 2$ , there is no  $q$ -uniformly smoothness Banach space. In [3], this assertion was showed by Cioranescu. We say that the mapping  $J$  is single-valued and also smoothness if the Banach space  $B$  having a sequentially continuous duality mapping  $J$  from weak topology to weak\* topology. The space  $B$  is said to have weakly sequentially continuous duality map if duality mapping  $J$  is continuous and single-valued, see [3, 4],

Let  $C$  be a nonempty subset of Banach space  $B$  and  $T : C \rightarrow B$  be a nonself mapping. Also, let  $F(T) = \{a \in C : Ta = a\}$  denote the set of fixed point of  $T$ . The map  $T : C \rightarrow B$  can be referred as follows:

- 1) It is nonexpansive if  $\|Ta - Tb\| \leq \|a - b\|$  for all  $a, b \in C$ .
- 2) It is quasi-nonexpansive if  $\|Ta - p\| \leq \|a - p\|$  for all  $a \in C$  and  $p \in F(T)$ .

In the following iterative process defined by Dogan and Karakaya [6].

Let  $C$  be a convex subset of a normed space  $B$  and  $T : C \rightarrow C$  a self map on  $B$ .

$$(1.1) \quad \begin{aligned} x_0 &= x \in C \\ f(T, x_n) &= (1 - \wp_n)x_n + \xi_n Tx_n + (\wp_n - \xi_n)Ty_n \\ y_n &= (1 - \zeta_n)x_n + \zeta_n Tx_n \end{aligned}$$

for  $n \geq 0$ , where  $\{\xi_n\}$ ,  $\{\wp_n\}$ ,  $\{\zeta_n\}$  satisfies the following conditions

- $C_1)$   $\wp_n \geq \xi_n$
- $C_2)$   $\{\wp_n - \xi_n\}_{n=0}^\infty, \{\wp_n\}_{n=0}^\infty, \{\zeta_n\}_{n=0}^\infty, \{\xi_n\}_{n=0}^\infty \in [0, 1]$
- $C_3)$   $\sum_{n=0}^\infty \wp_n = \infty$ .

In 1967, Halpern [7] was the first who introduced the following iteration process under the nonexpansive mapping  $T$ . For any initial value  $a_0 \in C$  and any fix  $u \in C$ ,  $\varphi_n \in [0, 1]$  such that  $\varphi_n = n^{-b}$ ,

$$(1.2) \quad a_{n+1} = \varphi_n u + (1 - \varphi_n)Ta_n \quad \forall n \in \mathbb{N},$$

where  $b \in (0, 1)$ . In 1977, Lions [8] showed that the iteration process (1.2) converges strongly to a fixed point of  $T$ , where  $\{\varphi_n\}_{n \in \mathbb{N}}$  provides the following first three conditions:

- $(C_1)$   $\lim_{n \rightarrow \infty} \varphi_n = 0$ ;
- $(C_2)$   $\sum_{n=1}^\infty \varphi_n = \infty$ ;
- $(C_3)$   $\lim_{n \rightarrow \infty} \frac{\varphi_{n+1} - \varphi_n}{\varphi_{n+1}^2} = 0$ ;
- $(C_4)$   $\sum_{n=1}^\infty |\varphi_{n+1} - \varphi_n| < \infty$ ;
- $(C_5)$   $\lim_{n \rightarrow \infty} \frac{\varphi_{n+1} - \varphi_n}{\varphi_{n+1}} = 0$ ;
- $(C_6)$   $|\varphi_{n+1} - \varphi_n| \leq o(\varphi_{n+1}) + \sigma_n, \sum_{n=1}^\infty \sigma_n < \infty$ .

Also, by exchanging of the above conditions, several authors were obtained various results in different spaces. Let us list the main ones as follows:

(1) In [9], Wittmann was shown that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  converges strongly of a fixed point of  $T$  by the conditions  $C_1$ ,  $C_2$  and  $C_4$ .

(2) In [10, 11], Reich was shown that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  converges strongly of a fixed point of  $T$  in the uniformly smooth Banach spaces by the conditions  $C_1$ ,  $C_2$  and  $C_6$ .

(3) In [12], Shioji and Takahashi were shown that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  converges strongly of a fixed point of  $T$  in the Banach spaces with uniformly Gâteaux differentiable norms by the conditions  $C_1$ ,  $C_2$  and  $C_4$ .

(4) In [13], Xu was shown that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  converges strongly of a fixed point of  $T$  by the conditions  $C_1$ ,  $C_2$  and  $C_5$ .

Are the conditions  $C_1$  and  $C_2$  enough to guarantee the strong convergence of (1.2) iteration process for the quasi-nonexpansive mappings, see [7]?

This question was answered positively by some authors. In the following list, you can see the work of these authors [14, 15, 16, 17, 18, 19, 20, 21]. But, in [22], they were shown that the answer to open question is not positive for nonexpansive mappings in Hilbert spaces.

The effective domain and range of  $A : B \rightarrow 2^B$  denoted by  $\text{dom}(A) = \{a \in B : Aa \neq \emptyset\}$  and  $R(A)$ , respectively. If there exists  $j \in J(a - b)$  such that  $\langle a - b, j \rangle \geq 0$  and  $J : B \rightarrow 2^{B^*}$  duality mapping, then the map  $A$  is said to be accretive, for all  $a, b \in B$ . If  $R(I + rA) = B$ , for each  $r \geq 0$ , then the accretive map  $A$  is

$m$ -accretive operator. All this paper, let  $A : B \rightarrow 2^B$  be an accretive operator and be has a zero. Now, we can define a single-valued mapping such that  $J_r = (I + rA)^{-1} : B \rightarrow \text{dom}(A)$ . It is called the resolvent of  $A$  for  $r > 0$ . Let  $A^{-1} = \{a \in B : 0 \in Aa\}$ . It is known that  $A^{-1} = F(J_r)$  for all  $r > 0$ , (see, [23, 24]).

Let  $B$  be a reflexive, smooth and strictly convex Banach space and  $C$  be a nonempty, closed and convex subset (ccs) of  $B$ . Under these conditions, for any  $a \in B$ , there exists a unique point  $z \in C$  such that

$$\|z - a\| \leq \min_{t \in C} \|t - a\|; \text{ see [24].}$$

**Definition 1.** [24] If  $P_C a = z$ , then the map  $P_C : B \rightarrow C$  is called the metric projection.

Assume that  $a \in B$  and  $z \in C$ , then  $z = P_C a$  iff  $\langle z - t, J(a - z) \rangle \geq 0$ , for all  $t \in C$ . In a real Hilbert space  $H$ , there is a  $P_C : H \rightarrow C$  projection mapping, which is nonexpansive, but, such a  $P_C : B \rightarrow C$  projection mapping does not provide the nonexpansive property in a Banach space  $B$ , where  $C$  is a nonempty, closed and convex subset of them; see [25].

**Definition 2.** [26] Let  $C \subset D$  be subsets of Banach space  $B$ . A mapping  $Q : C \rightarrow D$  is said to be a sunny if  $Q(\delta x + (1 - \delta)Qx) = Qx$ , for each  $x \in B$  and  $\delta \in [0, 1]$ .

$Q$  is said to be a retraction if and only if  $Q^2 = Q$ .  $Q$  is a sunny nonexpansive retraction if and only if it is sunny, nonexpansive and retraction.

In the next time, we will need lemmas in order to prove the main results.

**Lemma 1.** [13] Let  $B$  be a Banach space with weakly sequentially continuous duality mapping  $J_\phi$ . Then

$$\psi(\|a + b\|) \leq \psi(\|a\|) + 2\langle b, j_\phi(a + b) \rangle$$

for  $a, b \in B$ . If we get  $J$  instead of  $J_\phi$ , we have

$$\|a + b\|^2 \leq \|a\|^2 + 2\langle b, j(a + b) \rangle$$

for  $a, b \in B$ .

**Lemma 2.** [5] Let  $B$  be a Banach space with weakly sequentially continuous duality mapping  $J_\phi$  and  $C$  be a ccs of  $B$ . Let  $T : C \rightarrow C$  be a nonexpansive operator having  $F(T) \neq \emptyset$ . Then, for each  $u \in C$ , there exists  $a \in F(T)$  such that

$$\langle u - a, J(b - a) \rangle \leq 0$$

for all  $b \in F(T)$ .

**Lemma 3.** [27] Let  $B$  be a reflexive Banach space with weakly sequentially continuous duality mapping  $J_\phi$  and  $C$  be a ccs of  $B$ . Assume that  $T : C \rightarrow C$  is a nonexpansive operator. Let  $z_t \in C$  be the unique solution in  $C$  to the equation  $z_t = tu + (1 - t)Tz_t$  such that  $u \in C$  and  $t \in (0, 1)$ . Then  $T$  has a fixed point if and only if  $\{z_t\}_{t \in (0, 1)}$  remains bounded as  $t \rightarrow 0^+$ , and in this case,  $\{z_t\}_{t \in (0, 1)}$  converges as  $t \rightarrow 0^+$  strongly to fixed point of  $T$ . If we get the sunny nonexpansive retraction defined by  $Q : C \rightarrow F(T)$  such that

$$Q(u) = \lim_{t \rightarrow 0} z_t,$$

then  $Q(u)$  solves the variational inequality

$$\langle u - Q(u), J_\phi(b - Q(u)) \rangle \leq 0, \quad u \in C \text{ and } b \in F(T).$$

One of the useful and remarkable results in the theory of nonexpansive mappings is demiclosed principle. It is defined as follows.

**Definition 3.** [21] Let  $B$  be a Banach space,  $C$  a nonempty subset of  $B$ , and  $T : C \rightarrow B$  a mapping. Then the mapping  $T$  is said to be demiclosed at origin, that is, for any sequence  $\{a_n\}_{n \in \mathbb{N}}$  in  $C$  which  $a_n \rightharpoonup p$  and  $\|Ta_n - a_n\| \rightarrow 0$  imply that  $Tp = p$ .

**Lemma 4.** [28] Let  $B$  be a reflexive Banach space having weakly sequentially continuous duality mapping  $J_\phi$  with a gauge function  $\phi$ ,  $C$  be a ccs of  $B$  and  $T : C \rightarrow B$  be a nonexpansive mapping. Then  $I - T$  is demiclosed at each  $p \in B$ , i.e., for any sequence  $\{a_n\}_{n \in \mathbb{N}}$  in  $C$  which converges weakly to  $a$ , and  $(I - T)a_n \rightarrow p$  converges strongly imply that  $(I - T)a = p$ . (Here  $I$  is the identity operator of  $B$  into itself.) In particular, assuming  $p = 0$ , it is obtained  $a \in F(T)$ .

**Lemma 5.** [29] Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a nonnegative real sequence and satisfies the following inequality

$$\mu_{n+1} \leq (1 - \varphi_n) \mu_n + \varphi_n \epsilon_n,$$

and assume that  $\{\varphi_n\}_{n \in \mathbb{N}}$  and  $\{\epsilon_n\}_{n \in \mathbb{N}}$  satisfy the following conditions:

- (1)  $\{\varphi_n\}_{n \in \mathbb{N}} \subset [0, 1]$  and  $\sum_{n=1}^{\infty} \varphi_n = \infty$ ,
  - (2)  $\limsup_{n \rightarrow \infty} \epsilon_n \leq 0$ , or
  - (3)  $\sum_{n=1}^{\infty} \varphi_n \epsilon_n < \infty$ ,
- then  $\lim_{n \rightarrow \infty} \mu_n = 0$ .

**Lemma 6.** [24] Let  $B$  be a real Banach space, and let  $A$  be an  $m$ -accretive operator on  $B$ . For  $t > 0$ , let  $J_t$  be a resolvent operator related to  $A$  and  $t$ . Then

$$\|J_k a - J_l a\| \leq \frac{|k - l|}{k} \|a - J_k a\|, \text{ for all } k, l > 0 \text{ and } a \in B.$$

**Lemma 7.** [30] Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a sequence of real numbers such that there exists a subsequence  $\{\mu_{n_i}\}_{i \in \mathbb{N}}$  of  $\{\mu_n\}_{n \in \mathbb{N}}$  which satisfies  $\mu_{n_i} < \mu_{n_{i+1}}$  for all  $i \geq 0$ . Also, we consider a subsequence  $\{\eta_{(n)}\}_{n \geq n_0} \subset \mathbb{N}$  defined by

$$\eta_{(n)} = \max \{k \leq n : \mu_k \leq \mu_{k+1}\}.$$

Then  $\{\eta_{(n)}\}_{n \geq n_0}$  is a nondecreasing sequence providing  $\lim_{n \rightarrow \infty} \eta_{(n)} = \infty$ , for all  $n \geq n_0$ . Hence, it holds that  $\mu_{\eta_{(n)}} \leq \mu_{\eta_{(n)}+1}$ , and implies that  $\mu_n \leq \mu_{\eta_{(n)}+1}$ .

**Lemma 8.** [31] Let  $B$  be a uniformly convex Banach space and  $t > 0$  be a constant. Then there exists a continuous, strictly increasing and convex function  $g : [0, 2t) \rightarrow [0, \infty)$  such that

$$\left\| \sum_{i=1}^{\infty} \rho_i a_i \right\|^2 \leq \sum_{i=1}^{\infty} \rho_i \|a_i\|^2 - \rho_k \rho_l g(\|a_k - a_l\|)$$

$\forall k, l \geq 0, a_i \in B_t = \{z \in B : \|z\| \leq t\}, \rho_i \in (0, 1)$  and  $i \geq 0$  with  $\sum_{i=0}^{\infty} \rho_i = 1$ .

## 2. MAIN RESULTS

**Theorem 3.** *Let  $B$  be a real uniformly convex Banach space having the normalized duality mapping  $J$  and  $C$  be a ccs of  $B$ . Assume that  $\{T_i\}_{i \in \mathbb{N} \cup \{0\}}$  is a infinite family of quasi nonexpansive mappings given in the form  $T_i : C \rightarrow C$  such that  $F = \bigcap_{i=0}^{\infty} F(T_i) \neq \emptyset$ , and for each  $i \geq 0$ ,  $T_i - I$  is demiclosed at zero. Let  $\{v_n\}_{n \in \mathbb{N}}$  be a sequence generated by*

$$(2.1) \quad \begin{cases} v_1, u \in C \text{ arbitrarily chosen,} \\ v_{n+1} = \xi_n u + (1 - \xi_n) T_0 v_n + (\xi_n - \xi_n) T_0 w_n \\ w_n = \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} T_i v_n, \quad n \geq 0, \end{cases}$$

where  $\{\xi_n\}_{n \in \mathbb{N}}, \{\xi_n\}_{n \in \mathbb{N}}$  and  $\{\varphi_{n,i}\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  are sequences in  $[0, 1]$  satisfying the following control conditions:

- (1)  $\lim_{n \rightarrow \infty} \xi_n = 0$ ;
- (2)  $\sum_{n=1}^{\infty} \xi_n = \infty$ ;
- (3)  $\varphi_{n,0} + \sum_{i=1}^{\infty} \varphi_{n,i} = 1$ , for all  $n \in \mathbb{N}$ ;
- (4)  $\liminf_{n \rightarrow \infty} \xi_n \varphi_{n,0} \varphi_{n,i} > 0$ , for all  $n \in \mathbb{N}$ .

Then  $\{v_n\}_{n \in \mathbb{N}}$  converges strongly as  $n \rightarrow \infty$  to  $P_F u$ , where the map  $P_F : B \rightarrow F$  is the metric projection.

*Proof.* The proof consists of three parts.

Step 1. Prove that  $\{v_n\}_{n \in \mathbb{N}}, \{w_n\}_{n \in \mathbb{N}}$  and  $\{T_i v_n\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  are bounded. Firstly, we show that  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Let  $p \in F$  be fixed. By Lemma 8, we have the following inequality

$$(2.2) \quad \begin{aligned} \|w_n - p\|^2 &= \left\| \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} T_i v_n - p \right\|^2 \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|T_i v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \\ &= \|v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \\ &\leq \|v_n - p\|^2. \end{aligned}$$

This show that

$$\begin{aligned}
\|v_{n+1} - p\| &= \|\xi_n u + (1 - \zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \zeta_n) \|T_0 v_n - p\| + (\zeta_n - \xi_n) \|T_0 w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \zeta_n) \|v_n - p\| + (\zeta_n - \xi_n) \|w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \xi_n) \|v_n - p\| \\
&\leq \max \{ \|u - p\|, \|v_n - p\| \}
\end{aligned}$$

If we continue the way of induction, we have

$$\|v_{n+1} - p\| \leq \max \{ \|u - p\|, \|v_1 - p\| \}, \forall n \in \mathbb{N}.$$

Therefore, we conclude that  $\|v_{n+1} - p\|$  is bounded, this implies that  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Furthermore, it is easily show that  $\{T_i v_n\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  and  $\{w_n\}_{n \in \mathbb{N}}$  are bounded too.

Step 2. Show that for any  $n \in \mathbb{N}$ ,

$$(2.3) \quad \|v_{n+1} - z\|^2 \leq (1 - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J(v_{n+1} - z) \rangle.$$

By considering (2.2), we have

$$(2.4) \quad \|w_n - z\|^2 = \|v_n - z\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|).$$

(2.4) implies that

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \|\xi_n u + (1 - \zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n - z\|^2 \\
&\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \|T_0 v_n - z\|^2 + (\zeta_n - \xi_n) \|T_0 w_n - z\|^2 \\
(2.5) \quad &\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \|v_n - z\|^2 \\
(2.6) \quad &+ (\zeta_n - \xi_n) \left[ \|v_n - z\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \right] \\
&= \xi_n \|u - z\|^2 + (1 - \xi_n) \|v_n - z\|^2 \\
(2.7) \quad &- \zeta_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) + \xi_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|).
\end{aligned}$$

Assume that  $K_1 = \sup \left\{ \left| \|u - z\|^2 - \|v_n - z\|^2 \right| + \xi_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \right\}$ .

It is conclude form (2.5) that

$$(2.8) \quad \zeta_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - T_i v_n\|) \leq \|v_n - z\|^2 - \|v_{n+1} - z\|^2 + \xi_n K_1.$$

By Lemma 1 and (2.2), we have

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \|\xi_n u + (1 - \zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n - z\|^2 \\
&= \|\xi_n (u - z) + (1 - \zeta_n) (T_0 v_n - z) + (\zeta_n - \xi_n) (T_0 w_n - z)\|^2 \\
&\leq \|(1 - \zeta_n) (T_0 v_n - z) + (\zeta_n - \xi_n) (T_0 w_n - z)\|^2 \\
&\quad + 2 \langle \xi_n (u - z), J(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|T_0 v_n - z\|^2 + (\zeta_n - \xi_n) \|T_0 w_n - z\|^2 \\
&\quad + 2 \langle \xi_n (u - z), J(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|w_n - z\|^2 \\
&\quad + 2 \xi_n \langle u - z, J(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|v_n - z\|^2 \\
&\quad + 2 \xi_n \langle u - z, J(v_{n+1} - z) \rangle \\
&= (1 - \xi_n) \|v_n - z\|^2 + 2 \xi_n \langle u - z, J(v_{n+1} - z) \rangle.
\end{aligned}$$

Step 3. We show that  $v_n \rightarrow z$  as  $n \rightarrow \infty$ .

For this step, we will examine two cases.

Case 1. Suppose that there exists  $n_0 \in \mathbb{N}$  such that  $\{\|v_n - z\|\}_{n \geq n_0}$  is nonincreasing. furthermore, the sequence  $\{\|v_n - z\|\}_{n \in \mathbb{N}}$  is convergent. Thus, it is clear that  $\|v_n - z\|^2 - \|v_{n+1} - z\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . In view of condition (4) and (2.8), we have

$$\lim_{n \rightarrow \infty} g(\|v_n - T_i v_n\|) = 0.$$

From the properties of  $g$ , we have

$$\lim_{n \rightarrow \infty} \|v_n - T_i v_n\| = 0.$$

Also, we can construct the sequences  $(w_n - v_n)$  and  $(v_{n+1} - w_n)$ , as follows:

$$\begin{aligned} w_n - v_n &= \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} T_i v_n - v_n \\ (2.9) \quad &= \sum_{i=1}^{\infty} \varphi_{n,i} (T_i v_n - v_n), \end{aligned}$$

and

$$\begin{aligned} v_{n+1} - w_n &= \xi_n u + (1 - \zeta_n) T_0 v_n + (\zeta_n - \xi_n) T_0 w_n - w_n \\ (2.10) \quad \|v_{n+1} - w_n\| &= \|\xi_n (u - T_0 w_n) + \zeta_n (T_0 v_n - T_0 w_n) + (T_0 v_n - w_n)\| \\ &\leq \xi_n \|u - T_0 w_n\| + \zeta_n \|T_0 v_n - T_0 w_n\| + \|T_0 v_n - w_n\| \\ &\leq \xi_n \|u - T_0 w_n\| + \zeta_n \|v_n - w_n\| + \|T_0 v_n - w_n\|. \end{aligned}$$

These imply that

$$(2.11) \quad \lim_{n \rightarrow \infty} \|v_{n+1} - w_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|w_n - v_n\| = 0.$$

By the expressions in (2.11), we obtain

$$\|v_{n+1} - v_n\| \leq \|w_n - v_n\| + \|v_{n+1} - w_n\|.$$

This implies that

$$(2.12) \quad \lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0.$$

Previously, we have shown that the sequence  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Therefore, there exists a subsequence  $\{v_{n_j}\}_{j \in \mathbb{N}}$  of  $\{v_n\}_{n \in \mathbb{N}}$  such that  $v_{n_j+1} \rightarrow l$  for all  $j \in \mathbb{N}$ . By principle of demiclosedness at zero, It is concluded that  $l \in F$ . Considering the above facts and Definition (1), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, J(v_{n+1}, z) \rangle &= \lim_{j \rightarrow \infty} \langle u - z, J(v_{n_j+1} - z) \rangle \\ (2.13) \quad &= \langle u - z, J(l - z) \rangle \\ &= \langle u - P_F u, J(l - P_F u) \rangle \\ &\leq 0. \end{aligned}$$

By Lemma (5), we have the desired result.

Case 2. Let  $\{n_j\}_{j \in \mathbb{N}}$  be subsequence of  $\{n\}_{n \in \mathbb{N}}$  such that

$$\|v_{n_j} - z\| \leq \|v_{n_j+1} - z\|, \text{ for all } j \in \mathbb{N}.$$



Then, in view of Lemma (7), there exists a nondecreasing sequence  $\{m_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ , and hence

$$\|z - v_{m_k}\| < \|z - v_{m_k+1}\| \quad \text{and} \quad \|z - v_k\| \leq \|z - v_{m_k+1}\|, \quad \forall k \in \mathbb{N}.$$

If we rewrite the equation (2.8) for this Lemma, we have

$$\begin{aligned} \zeta_{m_k} \varphi_{m_k,0} \varphi_{m_k,i} g(\|v_{m_k} - T_i v_{m_k}\|) &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + \xi_{m_k} K_1 \\ &\leq \xi_{m_k} K_1, \quad \forall k \in \mathbb{N}. \end{aligned}$$

Considering the conditions (1) and (2), we obtain

$$\lim_{k \rightarrow \infty} g(\|v_{m_k} - T_i v_{m_k}\|) = 0.$$

It follows that

$$\lim_{k \rightarrow \infty} \|v_{m_k} - T_i v_{m_k}\| = 0.$$

Therefore, using the same argument as Case 1, we have

$$\limsup_{n \rightarrow \infty} \langle u - z, J(v_{m_k}, z) \rangle = \limsup_{n \rightarrow \infty} \langle u - z, J(v_{m_k+1}, z) \rangle \leq 0.$$

Using (2.3), we get

$$\|v_{m_k+1} - z\|^2 \leq (1 - \xi_{m_k}) \|v_{m_k} - z\|^2 + 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z) \rangle.$$

Previously, we have shown that the inequality  $\|v_{m_k} - z\| \leq \|v_{m_k+1} - z\|$  is performed, and hence

$$\begin{aligned} \xi_{m_k} \|v_{m_k} - z\|^2 &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z) \rangle \\ &\leq 2\xi_{m_k} \langle u - z, J(v_{m_k+1} - z) \rangle. \end{aligned}$$

Hence, we get

$$(2.14) \quad \lim_{k \rightarrow \infty} \|v_{m_k} - z\| = 0.$$

considering the expressions (2.13) and (2.14), we obtain

$$\lim_{k \rightarrow \infty} \|v_{m_k+1} - z\| = 0.$$

Finally, we get  $\|v_k - z\| \leq \|v_{m_k+1} - z\|$ ,  $\forall k \in \mathbb{N}$ . It follows that  $v_{m_k} \rightarrow z$  as  $k \rightarrow \infty$ . Then we have  $v_k \rightarrow z$  as  $n \rightarrow \infty$ .  $\square$

We obtain the following corollary for a single mapping.

**Corollary 1.** *Let  $B$  be a real uniformly convex Banach space having the normalized duality mapping  $J$  and  $C$  be a ccs of  $B$ . Assume that  $T$  is a quasi nonexpansive mappings given in the form  $T : C \rightarrow C$  and  $F$  is set of fixed point of  $T$  and,  $T - I$  is demiclosed at zero. Let  $\{v_n\}_{n \in \mathbb{N}}$  be a sequence generated by*

$$\begin{cases} v_1, u \in C \text{ arbitrarily chosen,} \\ v_{n+1} = \xi_n u + (1 - \zeta_n) T v_n + (\zeta_n - \xi_n) T w_n \\ w_n = (1 - \varphi_n) v_n + \varphi_n T v_n, \quad n \geq 0, \end{cases}$$

where  $\{\zeta_n\}_{n \in \mathbb{N}}$ ,  $\{\xi_n\}_{n \in \mathbb{N}}$  and  $\{\varphi_n\}_{n \in \mathbb{N}}$  are sequences in  $[0, 1]$  satisfying the following control conditions:

- (1)  $\lim_{n \rightarrow \infty} \xi_n = 0$ ;
- (2)  $\sum_{n=1}^{\infty} \xi_n = \infty$ ;
- (3)  $\liminf_{n \rightarrow \infty} \zeta_n (1 - \varphi_n) \varphi_n > 0$ , for all  $n \in \mathbb{N}$ .

Then  $\{v_n\}_{n \in \mathbb{N}}$  converges strongly as  $n \rightarrow \infty$  to  $P_F u$ , where the map  $P_F : B \rightarrow F$  is the metric projection.

**Theorem 4.** Let  $B$  be a real uniformly convex Banach space having the weakly sequentially continuous duality mapping  $J_\phi$  and  $C$  be a ccs of  $B$  such that  $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$  for each  $i \in N$ . Assume that  $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}$  is an infinite family of accretive operators satisfying the range condition, and  $r_n > 0$  and  $r > 0$  be such that  $\lim_{n \rightarrow \infty} r_n = r$ . Let  $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$  be the resolvent of  $A$ . Let  $\{v_n\}_{n \in \mathbb{N}}$  be a sequence generated by

$$(2.15) \quad \begin{cases} v_1, u \in C \text{ arbitrarily chosen,} \\ v_{n+1} = \xi_n u + (1 - \xi_n) J_{r_n}^{A_0} v_n + (\xi_n - \xi_n) J_{r_n}^{A_0} w_n \\ w_n = \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n, \quad n \geq 0, \end{cases}$$

where  $\{\xi_n\}_{n \in \mathbb{N}}$ ,  $\{\xi_n\}_{n \in \mathbb{N}}$  and  $\{\varphi_{n,i}\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  are sequences in  $[0, 1]$  satisfying the following control conditions:

- (1)  $\lim_{n \rightarrow \infty} \xi_n = 0$ ;
- (2)  $\sum_{n=1}^{\infty} \xi_n = \infty$ ;
- (3)  $\varphi_{n,0} + \sum_{i=1}^{\infty} \varphi_{n,i} = 1$ , for all  $n \in \mathbb{N}$ ;
- (4)  $\liminf_{n \rightarrow \infty} \xi_n \varphi_{n,0} \varphi_{n,i} > 0$ , for all  $n \in \mathbb{N}$ .

If  $Q_Z : B \rightarrow Z$  is the sunny nonexpansive retraction such that  $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ , then  $\{v_n\}_{n \in \mathbb{N}}$  converges strongly as  $n \rightarrow \infty$  to  $Q_Z u$ .

*Proof.* The proof consists of three parts.

We note that  $Z$  is closed and convex. Set  $z = Q_Z u$ .

Step 1. Prove that  $\{v_n\}_{n \in \mathbb{N}}$ ,  $\{w_n\}_{n \in \mathbb{N}}$  and  $\{J_{r_n}^{A_i} v_n\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  are bounded. Firstly, we show that  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Let  $p \in Z$  be fixed. By Lemma 8, we have the following inequality

$$\begin{aligned} \|w_n - p\|^2 &= \left\| \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n - p \right\|^2 \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|J_{r_n}^{A_i} v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \\ &\leq \varphi_{n,0} \|v_n - p\|^2 + \sum_{i=1}^{\infty} \varphi_{n,i} \|v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \\ &= \|v_n - p\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \\ (2.16) \quad &\leq \|v_n - p\|^2. \end{aligned}$$

This show that

$$\begin{aligned}
\|v_{n+1} - p\| &= \|\xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \zeta_n) \|J_{r_n}^{A_0} v_n - p\| + (\zeta_n - \xi_n) \|J_{r_n}^{A_0} w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \zeta_n) \|v_n - p\| + (\zeta_n - \xi_n) \|w_n - p\| \\
&\leq \xi_n \|u - p\| + (1 - \xi_n) \|v_n - p\| \\
&\leq \max \{ \|u - p\|, \|v_n - p\| \}
\end{aligned}$$

If we continue the way of induction, we have

$$\|v_{n+1} - p\| = \max \{ \|u - p\|, \|v_1 - p\| \}, \forall n \in \mathbb{N}.$$

Therefore, we conclude that  $\|v_{n+1} - p\|$  is bounded, this implies that  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Furthermore, it is easily show that  $\{J_{r_n}^{A_i} v_n\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  and  $\{w_n\}_{n \in \mathbb{N}}$  are bounded too.

Step 2. Show that for any  $n \in \mathbb{N}$ ,

$$(2.17) \quad \|v_{n+1} - z\|^2 \leq (1 - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J_\phi(v_{n+1} - z) \rangle.$$

By considering (2.16), we have

$$(2.18) \quad \|w_n - z\|^2 = \|v_n - z\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|).$$

(2.18) implies that

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \|\xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - z\|^2 \\
&\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \|J_{r_n}^{A_0} v_n - z\|^2 + (\zeta_n - \xi_n) \|J_{r_n}^{A_0} w_n - z\|^2 \\
&\leq \xi_n \|u - z\|^2 + (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \left[ \|v_n - z\|^2 - \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \right] \\
&= \xi_n \|u - z\|^2 + (1 - \xi_n) \|v_n - z\|^2 - \zeta_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) + \xi_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|)
\end{aligned}$$

Assume that  $K_2 = \sup \left\{ \left| \|u - z\|^2 - \|v_n - z\|^2 \right| + \xi_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \right\}$ .

It is conclude form (2.19) that

$$(2.20) \quad \zeta_n \varphi_{n,0} \varphi_{n,i} g(\|v_n - J_{r_n}^{A_i} v_n\|) \leq \|v_n - z\|^2 - \|v_{n+1} - z\|^2 + \xi_n K_2.$$

By Lemma 1 and (2.16), we have

$$\begin{aligned}
\|v_{n+1} - z\|^2 &= \|\xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - z\|^2 \\
&= \|\xi_n (u - z) + (1 - \zeta_n) (J_{r_n}^{A_0} v_n - z) + (\zeta_n - \xi_n) (J_{r_n}^{A_0} w_n - z)\|^2 \\
&\leq \|(1 - \zeta_n) (J_{r_n}^{A_0} v_n - z) + (\zeta_n - \xi_n) (J_{r_n}^{A_0} w_n - z)\|^2 + 2 \langle \xi_n (u - z), J_\phi(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|J_{r_n}^{A_0} v_n - z\|^2 + (\zeta_n - \xi_n) \|J_{r_n}^{A_0} w_n - z\|^2 + 2 \langle \xi_n (u - z), J_\phi(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|w_n - z\|^2 + 2\xi_n \langle u - z, J_\phi(v_{n+1} - z) \rangle \\
&\leq (1 - \zeta_n) \|v_n - z\|^2 + (\zeta_n - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J_\phi(v_{n+1} - z) \rangle \\
&= (1 - \xi_n) \|v_n - z\|^2 + 2\xi_n \langle u - z, J_\phi(v_{n+1} - z) \rangle.
\end{aligned}$$

Step 3. We show that  $v_n \rightarrow z$  as  $n \rightarrow \infty$ .

For this step, we will examine two cases.

Case 1. Suppose that there exists  $n_0 \in \mathbb{N}$  such that  $\{\|v_n - z\|\}_{n \geq n_0}$  is nonincreasing. furthermore, the sequence  $\{\|v_n - z\|\}_{n \in \mathbb{N}}$  is convergent. Thus, it is clear

that  $\|v_n - z\|^2 - \|v_{n+1} - z\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . In view of condition (4) and (2.20), we have

$$\lim_{n \rightarrow \infty} g(\|v_n - J_{r_n}^{A_i} v_n\|) = 0.$$

From the properties of  $g$ , we have

$$\lim_{n \rightarrow \infty} \|v_n - J_{r_n}^{A_i} v_n\| = 0.$$

Also, we can construct the sequences  $(w_n - v_n)$  and  $(v_{n+1} - w_n)$ , as follows:

$$\begin{aligned} w_n - v_n &= \varphi_{n,0} v_n + \sum_{i=1}^{\infty} \varphi_{n,i} J_{r_n}^{A_i} v_n - v_n \\ (2.21) \quad &= \sum_{i=1}^{\infty} \varphi_{n,i} (J_{r_n}^{A_i} v_n - v_n), \end{aligned}$$

and

$$\begin{aligned} v_{n+1} - w_n &= \xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n - w_n \\ \|v_{n+1} - w_n\| &= \|\xi_n (u - w_n) + (1 - \zeta_n) (J_{r_n}^{A_0} v_n - w_n) + (\zeta_n - \xi_n) (J_{r_n}^{A_0} w_n - w_n)\| \\ &\leq \xi_n \|u - w_n\| + (1 - \zeta_n) \|J_{r_n}^{A_0} v_n - w_n\| + (\zeta_n - \xi_n) \|J_{r_n}^{A_0} w_n - w_n\|. \end{aligned}$$

These imply that

$$(2.22) \quad \lim_{n \rightarrow \infty} \|v_{n+1} - w_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|w_n - v_n\| = 0.$$

By the expressions in (2.22), we obtain

$$\|v_{n+1} - v_n\| \leq \|w_n - v_n\| + \|v_{n+1} - w_n\|.$$

This implies that

$$(2.23) \quad \lim_{n \rightarrow \infty} \|v_{n+1} - v_n\| = 0.$$

By Lemma 6 and (2.21), we have

$$\|v_n - J_r^{A_i} v_n\| \leq \|v_n - J_{r_n}^{A_i} v_n\| + \|J_{r_n}^{A_i} v_n - J_r^{A_i} v_n\| \leq \|v_n - J_{r_n}^{A_i} v_n\| + \frac{|r_n - r|}{r_n} \|v_n - J_{r_n}^{A_i} v_n\|.$$

This implies that

$$\lim_{n \rightarrow \infty} \|v_n - J_r^{A_i} v_n\| = 0, \text{ for all } i \in \mathbb{N}.$$

Previously, we have shown that the sequence  $\{v_n\}_{n \in \mathbb{N}}$  is bounded. Therefore, there exists a subsequence  $\{v_{n_j}\}_{j \in \mathbb{N}}$  of  $\{v_n\}_{n \in \mathbb{N}}$  such that  $v_{n_j+1} \rightarrow l \in F(J_r^{A_i} v_n)$  for all  $j \in \mathbb{N}$ . This, together with Lemma 1 implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, J_\phi(v_{n+1}, z) \rangle &= \lim_{k \rightarrow \infty} \langle u - z, J_\phi(v_{n_j+1} - z) \rangle \\ (2.24) \quad &= \langle u - z, J_\phi(l - z) \rangle \\ &\leq 0. \end{aligned}$$

By Lemma (5), we obtain the desired result.

Case 2. Let  $\{n_j\}_{j \in \mathbb{N}}$  be subsequence of  $\{n\}_{n \in \mathbb{N}}$  such that

$$\|v_{n_j} - z\| \leq \|v_{n_j+1} - z\|, \text{ for all } j \in \mathbb{N}.$$

Then, in view of Lemma (7), there exists a nondecreasing sequence  $\{m_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ , and hence

$$\|z - v_{m_k}\| < \|z - v_{m_k+1}\| \quad \text{and} \quad \|z - v_k\| \leq \|z - v_{m_k+1}\|, \quad \forall k \in \mathbb{N}.$$

If we rewrite the equation (2.8) for this Lemma, we have

$$\begin{aligned} \zeta_{m_k} \varphi_{m_k,0} \varphi_{m_k,i} g(\|v_{m_k} - J_{r_n}^{A_i} v_{m_k}\|) &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + \xi_{m_k} K_2 \\ &\leq \xi_{m_k} K_2, \quad \forall k \in \mathbb{N}. \end{aligned}$$

Considering the conditions (1) and (2), we obtain

$$\lim_{k \rightarrow \infty} g(\|v_{m_k} - J_{r_n}^{A_i} v_{m_k}\|) = 0.$$

It follows that

$$\lim_{k \rightarrow \infty} \|v_{m_k} - J_{r_n}^{A_i} v_{m_k}\| = 0.$$

Therefore, using the same argument as Case 1, we have

$$\limsup_{n \rightarrow \infty} \langle u - z, J_\phi(v_{m_k}, z) \rangle = \limsup_{n \rightarrow \infty} \langle u - z, J_\phi(v_{m_k+1}, z) \rangle \leq 0.$$

Using (2.17), we get

$$\|v_{m_k+1} - z\|^2 \leq (1 - \xi_{m_k}) \|v_{m_k} - z\|^2 + 2\xi_{m_k} \langle u - z, J_\phi(v_{m_k+1} - z) \rangle.$$

Previously, we have shown that the inequality  $\|v_{m_k} - z\| \leq \|v_{m_k+1} - z\|$  is performed, and hence

$$\begin{aligned} \xi_{m_k} \|v_{m_k} - z\|^2 &\leq \|v_{m_k} - z\|^2 - \|v_{m_k+1} - z\|^2 + 2\xi_{m_k} \langle u - z, J_\phi(v_{m_k+1} - z) \rangle \\ &\leq 2\xi_{m_k} \langle u - z, J_\phi(v_{m_k+1} - z) \rangle. \end{aligned}$$

Hence, we get

$$(2.25) \quad \lim_{k \rightarrow \infty} \|v_{m_k} - z\| = 0.$$

Considering the expressions (2.24) and (2.25), we obtain

$$\lim_{k \rightarrow \infty} \|v_{m_k+1} - z\| = 0.$$

Finally, we get  $\|v_k - z\| \leq \|v_{m_k+1} - z\|$ ,  $\forall k \in \mathbb{N}$ . It follows that  $v_{m_k} \rightarrow z$  as  $k \rightarrow \infty$ . Then we have  $v_k \rightarrow z$  as  $n \rightarrow \infty$ .  $\square$

**Theorem 5.** Let  $B$  be a real uniformly convex Banach space having a Gâteaux differentiable norm. and  $C$  be a ccs of  $B$  such that  $\overline{D(A_i)} \subset C \subset \bigcap_{r>0}^\infty R(I + rA_i)$  for each  $i \in N$ . Assume that  $\{A_i\}_{i \in \mathbb{N} \cup \{0\}}$  is an infinite family of accretive operators satisfying the range condition, and  $r_n > 0$  and  $r > 0$  be such that  $\lim_{n \rightarrow \infty} r_n = r$ . Let  $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$  be the resolvent of  $A$ . Let  $\{v_n\}_{n \in \mathbb{N}}$  be a sequence generated by

$$(2.26) \quad \begin{cases} v_1, u \in C \text{ arbitrarily chosen,} \\ v_{n+1} = \xi_n u + (1 - \zeta_n) J_{r_n}^{A_0} v_n + (\zeta_n - \xi_n) J_{r_n}^{A_0} w_n \\ w_n = \varphi_{n,0} v_n + \sum_{i=1}^\infty \varphi_{n,i} J_{r_n}^{A_i} v_n, \quad n \geq 0, \end{cases}$$

where  $\{\zeta_n\}_{n \in \mathbb{N}}$ ,  $\{\xi_n\}_{n \in \mathbb{N}}$  and  $\{\varphi_{n,i}\}_{n \in \mathbb{N}, i \in \mathbb{N} \cup \{0\}}$  are sequences in  $[0, 1]$  satisfying the following control conditions:

$$(1) \quad \lim_{n \rightarrow \infty} \xi_n = 0;$$

- (2)  $\sum_{n=1}^{\infty} \xi_n = \infty$ ;
- (3)  $\varphi_{n,0} + \sum_{i=1}^{\infty} \varphi_{n,i} = 1$ , for all  $n \in \mathbb{N}$ ;
- (4)  $\liminf_{n \rightarrow \infty} \zeta_n \varphi_{n,0} \varphi_{n,i} > 0$ , for all  $n \in \mathbb{N}$ .

If  $Q_Z : B \rightarrow Z$  is the sunny nonexpansive retraction such that  $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$ , then  $\{v_n\}_{n \in \mathbb{N}}$  converges strongly as  $n \rightarrow \infty$  to  $Q_Z u$ .

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